STABILITY OF A RECTANGULAR PLATE BEYOND THE ELASTIC LIMIT

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The problem is resolved for buckling of a hinged elongated rectangular plate compressed along the long side from the position of two new plasticity theories [1, 2]. It is assumed that the start of buckling is accompanied by an increase in compressive load.

Theoretical and experimental studies devoted to this problem show that deformation theory (Nadai, Il'yushin) gives better conformity with experiments than flow theory (Reis, Laning) [3]. On the other hand, it is well known that deformation theory can only reasonably be used with proportional loading, whereas flow theory does not have such limitiations.

In the process of loss of stability for a plate there is complex loading with rotation of the principal stress axes. Each element of a material initially loaded proportionally at the instant of buckling is additionally loaded with application of all stress components under conditions of a plane stressed state. Recently this class of complex loads has been studied by experiment [4-6], and it has been established that neither deformation theory nor flow theory are suitable for quantitative description of these tests.

Illustrated below is the possibility of using the theories in [1, 2] for different directional complex additional loading from a stressed state. A comparison is provided with test data in [6]. The good agreement obtained of calculation and experiment makes it possible to use it as a better basis for solving stability probelms [1, 2] than deformation and flow theories. Results of calculations for the stability of a plate are compared with tests in [3, 7].

1. The most complete experiments presented in the literature are those in which different complex additional loading is accomplished from a state of uniaxial tension [4-6]. We return to analyze them with the aim of determining material characteristics used in the theories in [1, 2].

As a rule experiments for complex loading with rotation of the principal stress axes are performed on thin-walled tubes with application of an axial force and a torsional moment. Let axis x be directed along the generating line, and axis y have a tangential direction. Stress components differing from zero are designated σ_x and τ_{xy}. Arbitrary additional loading may be characterized by the parameter $m = \Delta\sigma_x/\Delta\tau_{xy}$.

From the position of the theories in [1, 2] we consider for a plastically incompressible material different additional loading from a stressed state by means of applying $\Delta\sigma_x$, $\Delta\tau_{xy}$. On the basis of [1] with tension we have

$$\Delta\varepsilon_x = \left(\frac{1+\sin\varphi}{2G_0} + \frac{1}{E}\right)\Delta\sigma_x, \quad \Delta\varepsilon_y = \Delta\varepsilon_z = -\frac{\Delta\varepsilon_x}{2} + \frac{1-2\nu}{2E}\Delta\sigma_x$$

(E is Young's modulus, ν is Poisson's ration, φ is internal friction angle). From the tangential modulus known from a test $E_t = \Delta\sigma_x/\Delta\varepsilon_x$ it is possible to determine the plastic strengthening modulus G_0

$$\frac{E}{G_0} = \frac{2}{1+\sin\varphi}\left(\frac{E}{E_t} - 1\right)$$

(G_0 and φ with uniaxial loading are functions of σ_x [1]).

The region of active additional loading when positive plastic displacements are realized through the whole slip plane is determined by the conditions $\Delta\sigma_x > 0$, $-(1+\sin\varphi)/(2\sin 2\varphi) \leqslant$

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TABLE 1

Specimen number	m	$G_i \cdot 10^{-2}$ MPa		
		experiment	theory in [1]	theory in [2]
A-1	—0,28	120	111	114
A-2	—0,40	117	120	125
A-3	—0,29	113	112	115
A-4	—0,32	116	114	118
A-5	—0,40	127	120	125
B-1	—2,0	232	237	237
C-1	—0,06	113	98	98
C-2	—0,025	95	96	96
D-1	1,2	84	59	61
D-2	1,1	77	60	62

$m^{-1} \leqslant (1 + \sin\varphi)/(2\sin 2\varphi)$. In this case

$$\Delta\varepsilon_x = \left(\frac{1+\sin\varphi}{2G_0} + \frac{1}{E}\right)\Delta\sigma_x, \quad \Delta\gamma_{xy} = \left(\frac{2\sin^2\varphi}{G_0} + \frac{1}{\mu}\right)\Delta\tau_{xy} \tag{1.1}$$

(μ is elastic shear modulus, $\Delta\gamma_{xy}$ is shear strain increment).

If $\Delta\tau_{xy} > 0,\ 0 \leqslant m \leqslant 2\sin 2\varphi/(1+\sin\varphi)$, then one of the slip planes is unloaded and the following relationships are valid

$$\begin{aligned} \Delta\varepsilon_x &= \left(\frac{3(1+\sin\varphi)}{8G_0} + \frac{1}{E}\right)\Delta\sigma_x + \frac{\sin 2\varphi}{4G_0}\Delta\tau_{xy}, \\ \Delta\gamma_{xy} &= \frac{1+\sin\varphi}{4G_0}\operatorname{tg}\varphi\Delta\sigma_x + \left(\frac{\sin^2\varphi}{G_0} + \frac{1}{\mu}\right)\Delta\tau_{xy}. \end{aligned} \tag{1.2}$$

With $\Delta\tau_{xy} > 0,\ -2\sin 2\varphi/(1+\sin\varphi) \leqslant m \leqslant 0$ partial unloading is realized, for which

$$\begin{aligned} \Delta\varepsilon_x &= \left(\frac{1+\sin\varphi}{8G_0} + \frac{1}{E}\right)\Delta\sigma_x + \frac{\sin 2\varphi}{4G_0}\Delta\tau_{xy}, \\ \Delta\gamma_{xy} &= \frac{1+\sin\varphi}{4G_0}\operatorname{tg}\varphi\Delta\sigma_x + \left(\frac{\sin^2\varphi}{G_0} + \frac{1}{\mu}\right)\Delta\tau_{xy}. \end{aligned} \tag{1.3}$$

If with additional loading from tension $\Delta\sigma_x < 0,\ -(1+\sin\varphi)/(2\sin 2\varphi) \leqslant m^{-1} \leqslant (1+\sin\varphi)/(2\sin 2\varphi)$, then there is unloading throughout the whole slip system and in this region the material deforms elastically.

As can be seen from (1.1)-(1.3), strains $\Delta\varepsilon_x$, $\Delta\gamma_{xy}$ change continuously in relation to the additional loading direction m.

We turn to the results of experiments for complex loading. Good conformity of calculated dependences for a model [1] with test data for St. 20 [4] is obtained with $\varphi = 0.12$. With an increase in the stress level angle φ increased [5]: $\varphi = 0.15$–0.41 (for brass), 0.18 (for duralumin), 0.3 (for copper).

In tests [6] on thin-walled tubular specimens made of aluminum alloy 24S-T4 a study was made of the dependence of initial modulus $G_i = \Delta\tau_{xy}/\Delta\gamma_{xy}$ on additional loading direction m. We shall have good conformity of theoretical and experimental results if we assume that G_0 = 3940 MPa, $\varphi = \pi/6$.

Proceeding from the theory in [1], modulus G_0 may also be determined by E_k in compression known from a test:

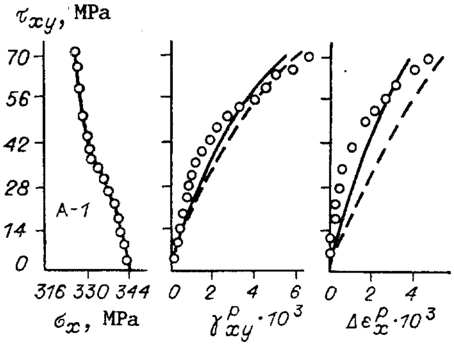

Fig. 1

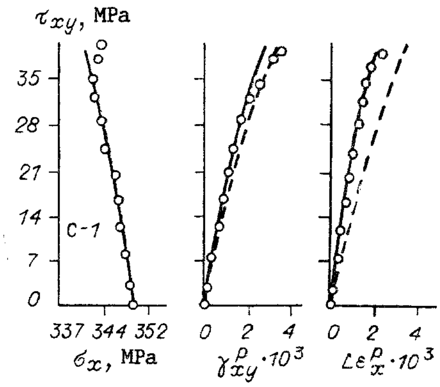

Fig. 2

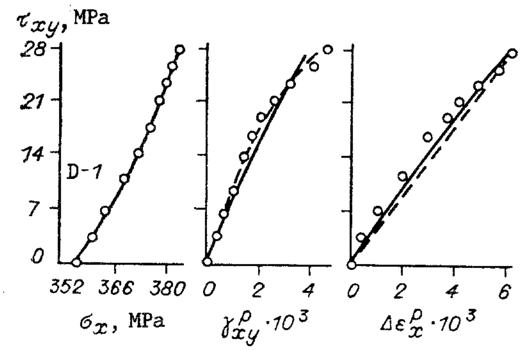

Fig. 3

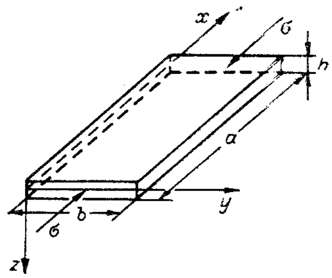

Fig. 4

$$\frac{E}{G_0} = \frac{2}{1-\sin\varphi}\left(\frac{E}{E_t} - 1\right). \tag{1.4}$$

With a plane stressed state in direction z $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. In this case relationships [1] for the region of active additional loading from a compressed state in direction x take the form

$$\begin{aligned} \Delta\varepsilon_x &= \left(\frac{1-\sin\varphi}{2G_0} + \frac{1}{E}\right)\Delta\sigma_x - \left(\frac{1+\sin\varphi}{4G_0} + \frac{\nu}{E}\right)\Delta\sigma_y, \\ \Delta\varepsilon_y &= -\left(\frac{1-\sin\varphi}{4G_0} + \frac{\nu}{E}\right)\Delta\sigma_x + \left(\frac{1+\sin\varphi}{4G_0} + \frac{1}{E}\right)\Delta\sigma_y, \\ \Delta\gamma_{xy} &= \left(\frac{2\sin^2\varphi}{G_0} + \frac{1}{\mu}\right)\Delta\tau_{xy}. \end{aligned} \tag{1.5}$$

Fundamental relationships of the theory in [2] for tension and compression are identical and they conform with Reis–Laning flow theory based on an isotropically expanding Mises surface:

$$\Delta\varepsilon_x = \left(\frac{1}{G_t} + \frac{1-2\nu}{E}\right)\frac{\Delta\sigma_x}{3}, \quad \Delta\varepsilon_y = \Delta\varepsilon_z = -\frac{\Delta\varepsilon_x}{2} + \frac{1-2\nu}{2E}\Delta\sigma_x. \tag{1.6}$$

As can be seen from (1.6), from the tangential modulus E_t known from a test it is possible to find strengthening modulus G_t:

$$E/G_t = 3E/E_t - 1 + 2\nu.$$

In solving the problem for buckling of a rectangular plate beyond the elastic limit we use equations of the theory in [2] applied to a plane stressed state in direction z. In this case for the region of active additional loading from a uniaxial compressed state in direction x definitive relationships take the form

$$\Delta\varepsilon_x = \left(\frac{1-2\nu}{3E} + \frac{1}{3G_t}\right)\Delta\sigma_x + \left(\frac{1-2\nu}{3E} - \frac{1}{6G_t}\right)\Delta\sigma_y,$$
$$\Delta\varepsilon_y = \left(\frac{1-2\nu}{3E} - \frac{1}{6G_t}\right)\Delta\sigma_x + \left(\frac{5-\nu}{6E} + \frac{1}{12G_t}\right)\Delta\sigma_y, \quad \Delta\gamma_{xy} = \left(\frac{2\delta}{3G_t} + \frac{1-2\delta/3}{\mu}\right)\Delta\tau_{xy} \tag{1.7}$$

(G_k and δ are functions of tangential stress intensity determined on the basis of test data). In [2] different complex additional loadings from a uniaxial tensile state with application of $\Delta\sigma_x$, $\Delta\tau_{xy}$ are considered in detail.

We analyze the data of experiments in [4-6] from the position of plasticity theory [2]. There is better conformity of calculated dependences as a result of experiments on St. 20 [4] with $\delta = 0.026$. As far as experiments in [5] are concerned, the entirely satisfactory conformity of calculation and experiment is only obtained for the first group of tests (additional loading by torsion). It has been established [2] that with an increase in σ_x values of δ increase: for brass $\delta = 0.1$-0.47, for duralumin it is 0.15-0.45. As follows from the theory in [2], additional loading with tension from a torsional state should not cause a change in plastic deformation, but this is not confirmed in tests in [5].

Now we turn to the results of experiments in [6]. There is better conformity of theoretical and test data with $G_t = 1640$ MPa, $\delta = 0.33$. Presented in Table 1 are test data for the initial additional loading modulus G_i as a result of calculations by the theory in [1] and that in [2]. Test data (circles) and calculated curves $\gamma_{xy}^p(\tau_{xy})$, $\Delta\varepsilon_x^p(\tau_{xy})$ by the theories in [1] (solid line) and in [2] (broken line) are provided in Figs. 1-3 for sections of complex loading. Also shown there are programs for loadings $\sigma_x(\tau_{xy})$.

2. We apply the definitive relationships for the theories considered above in solving the problem of buckling of hinged elongated rectangular plate compressed along the long side.

The first studies for buckling of plates and shells on the basis of deformation plasticity theory and a classical ideal about the loss of stability with unchanged external forces belong to Il'yushin [8] et al. In this approach buckling is accompanied by appearance of areas of unloading which markedly complicate the analysis. In using flow theory and the same criteria some of the difficulties are retained.

These problems are resolved much more simply if in proceeding from some plasticity theory the lower critical load is found corresponding to buckling with continuous loading (Shenley, 1946-1947). In [3] we find similar references to test data for stability of plates beyond the elastic limit and in works in which analysis of stability is performed using this criterion from the position of deformation theory and flow theory.

We proceed to solving the problem of buckling of a rectangular plate from the position of the theories in [1, 2] using the criterion of continuous loading.

A plate of constant thickness h (Fig. 4) with sides a and b (a >> b) before the instant of buckling is in a uniform stressed state $\sigma_x = -\sigma$, $\sigma_y = \tau_{xy} = 0$.

We write fundamental relationships for the region of active additional loading from the condition of uniaxial compression in a general form

$$\Delta\sigma_x = E(a_{11}\Delta\varepsilon_x + a_{12}\Delta\varepsilon_y),$$
$$\Delta\sigma_y = E(a_{21}\Delta\varepsilon_x + a_{22}\Delta\varepsilon_y), \quad \Delta\tau_{xy} = Ea_{33}\Delta\gamma_{xy}. \tag{2.1}$$

According to the Kirchhoff plate bending theory for buckling strain increments — linear functions of the distance from the central surface.

$$\Delta\varepsilon_x = \Delta\varepsilon_x^0 - z\partial^2 w/\partial x^2,$$
$$\Delta\varepsilon_y = \Delta\varepsilon_y^0 - z\partial^2 w/\partial y^2, \quad \Delta\gamma_{xy} = \Delta\gamma_{xy}^0 - 2z\partial^2 w/(\partial x\,\partial y), \tag{2.2}$$

where $\Delta\varepsilon_x^0$, $\Delta\varepsilon_y^0$, $\Delta\gamma_{xy}^0$ are infinitely small strain increments for the central surface; w = w(x, y) is plate deflection with buckling.

Apparently stress increments also change linearly through the plate thickness. Now by using (2.1) and (2.2) it is easy to calculate increments for bending ΔM_x, ΔM_y and torsional ΔM_{xy} moments:

$$\Delta M_x = \int_{-h/2}^{h/2} \Delta\sigma_x z dz = -\frac{Eh^3}{12}\left(a_{11}\frac{\partial^2 w}{\partial x^2} + a_{12}\frac{\partial^2 w}{\partial y^2}\right),$$

$$\Delta M_y = \int_{-h/2}^{h/2} \Delta\sigma_y z dz = -\frac{Eh^3}{12}\left(a_{21}\frac{\partial^2 w}{\partial x^2} + a_{22}\frac{\partial^2 w}{\partial y^2}\right), \quad (2.3)$$

$$\Delta M_{xy} = \int_{-h/2}^{h/2} \Delta\tau_{xy} z dz = -\frac{Eh^3}{6} a_{33}\frac{\partial^2 w}{\partial x \partial y}.$$

By projecting on the z axis forces operating on an element of the plate in a curved state, we obtain the well-known equilibrium equation [3]

$$\frac{\partial^2 \Delta M_x}{\partial x^2} + 2\frac{\partial^2 \Delta M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 \Delta M_y}{\partial y^2} + \sigma_x h \frac{\partial^2 w}{\partial x^2} = 0. \quad (2.4)$$

By substituting (2.3) in (2.4) we find with $\sigma_x = -\sigma$ a differential equation for plate buckling

$$a_{11}\frac{\partial^4 w}{\partial x^4} + (a_{12} + a_{21} + 4a_{33})\frac{\partial^4 w}{\partial x^2 \partial y^2} + a_{22}\frac{\partial^4 w}{\partial y^4} - \frac{12\sigma}{Eh^2}\frac{\partial^2 w}{\partial x^2} = 0. \quad (2.5)$$

For a plate hinged along the long edges we have the following boundary conditions with $y = 0,\ b$: $w = 0$, $\partial^2 w/\partial y^2 = 0$. We take a solution which satisfies these conditions in the form

$$w = A \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \quad (2.6)$$

By substituting this expression for deflection in (2.5) and defining $\lambda = a^2/(bm)^2$, we find the condition for existence of a nontrivial solution in the form of (2.6)

$$\sigma = \frac{Eh^2\pi^2}{12b^2}\left[\frac{a_{11}}{\lambda^2} + (a_{12} + a_{21} + 4a_{33})n^2 + a_{22}n^4\lambda^2\right].$$

It is easy to see that in order to find the least value of $\sigma = \sigma_*$ it is necessary to set $n = 1$, $\lambda^4 = a_{11}/a_{22}$, then

$$\sigma_* = \frac{E\pi^2 h^2}{12b^2}\left(2\sqrt{a_{11}a_{22}} + a_{12} + a_{21} + 4a_{33}\right). \quad (2.7)$$

Thus, (2.7) determines the least compressive load with which there is buckling of an elongated hinged rectangular plate compressed along the long side.

For an elastic material coefficients in relationship (2.1) take the form

$$a_{11} = a_{22} = \frac{1}{1-\nu^2}, \quad a_{12} = a_{21} = \frac{\nu}{1-\nu^2}, \quad a_{33} = \frac{1}{2(1+\nu)}. \quad (2.8)$$

By substituting (2.8) in (2.7) we find for an elastic region

$$\sigma_* = \frac{E\pi^2}{3(1-\nu^2)}\left(\frac{h}{b}\right)^2.$$

This equation is only valid in the case when the critical stress is within the elastic limits. By proceeding from relationship (1.5) we find coefficients in (2.1) for the plastic region of active loading

$$\begin{gathered}
a_{11} = \left(\frac{1+\sin\varphi}{4G_0}E+1\right)\Big/D, \quad a_{12} = \left(\frac{1+\sin\varphi}{4G_0}E+\nu\right)\Big/D, \\
a_{21} = \left(\frac{1-\sin\varphi}{4G_0}E+\nu\right)\Big/D, \quad a_{22} = \left(\frac{1-\sin\varphi}{2G_0}E+1\right)\Big/D, \\
a_{33}^{-1} = 2\sin^2\varphi E/G_0 + 2(1+\nu), \\
D = \frac{\cos^2\varphi}{16}\left(\frac{E}{G_0}\right)^2 + \frac{3-2\nu-\sin\varphi}{4G_0}E + 1 - \nu^2.
\end{gathered} \tag{2.9}$$

By substituting (2.9) in (2.7) we derive the dependence of σ_* on b/h for the theory in [1].

If now we resolve the relative increment of stresses of relationship (1.7) in the form of (2.1), then we obtain

$$\begin{gathered}
a_{11} = \frac{1+2(5-\nu)G_t/E}{D}, \quad a_{22} = 4\frac{1+(1-2\nu)G_t/E}{D}, \\
a_{12} = a_{21} = 2\frac{1-2(1-2\nu)G_t/E}{D}, \\
a_{33} = \frac{G_t/E}{2\delta(1-G_t/\mu)/3+G_t/\mu}, \\
D = 5 - 4\nu + (1-2\nu)G_t/\mu.
\end{gathered} \tag{2.10}$$

We compare calculated dependences σ_* (b/h) with the results of experiments in [7]. In [3] on the basis of test data for duralumin plates with elastically clamped edges [7] the dependence σ_*(b/h) is converted to the case of hinged edges. Results of these tests are shown in Fig. 5 by circles. The calculated dependence $\sigma_* = \sigma_*(b/h)$ according to the theory in [1] is applied by a solid line. In the calculations E = 75,000 MPa, ν = 0.3. In (2.9) with σ_* < 340 MPa angle φ = 0, then with σ_* = 340 MPa φ = 0.052(3°), with σ_* = 356 MPa φ = 0.105(6°), and with σ_* = 364 MPa φ = 0.156(9°). The metal used in the tests is similar to duralumin D16T, and therefore the plastic strengthening modulus G_0 was determined from (1.4) and the results of compression experiments on specimens of D16T [3].

In plotting calculated dependences from the theory in [2] the strengthening modulus G_k was found from the tangential modulus E_k known from uniaxial compression tests [3]. If in (2.10) we take δ = 0 and we substitute it in (2.7), then we obtain a calculation relation-

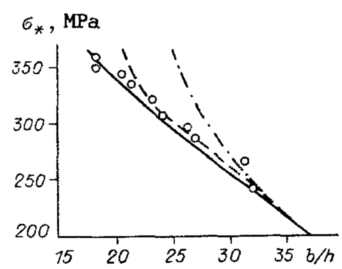

Fig. 5

ship $\sigma_* = \sigma_*(b/h)$ for flow theory with an isotropically expanding Mises surface which is shown in Fig. 5 by a broken-dotted line. The best conformity with the experiments may be given as a result of an increase in δ. It follows from analyzing test data in [5] that for duralumin D16T the maximum value of $\delta \approx 0.5$. The calculation relationship $\sigma_* = \sigma_*(b/h)$, calculated from the theory in [2] with $\delta = 0.5$ is given by a broken line in Fig. 5.

On the basis of the comparison provided it is possible to conclude that in solving the problem considered above for stability of a plate preference should be given to the theory in [1].

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